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Abstract A model of Poissonian observation having a jump (change-point) in the intensity function is considered. Two cases are studied. The first one corresponds to the situation when the jump size converges to a non-zero limit, while in the second one the limit is zero. The limiting likelihood ratios in these two cases are quite different. In the first case, like in the case of a fixed jump size, the normalized likelihood ratio converges to a log Poisson process. In the second case, the normalized likelihood ratio converges to a log Wiener process, and so, the statistical problems of parameter estimation and hypotheses testing are asymptotically equivalent in this case to the well known problems of change-point estimation and testing for the model of a signal in white Gaussian noise. The properties of the maximum likelihood and Bayesian estimators, as well as those of the general likelihood ratio, Wald's and Bayesian tests are deduced form the convergence of normalized likelihood ratios. The convergence of the moments of the estimators is also established. The obtained theoretical results are illustrated by numerical simulations.

Keywords Poisson process · non-regularity · change-point · limiting likelihood ratio process · maximum likelihood estimator · Bayesian estimators · consistency · limiting distribution · efficiency · general likelihood ratio test · Wald's test · Bayesian tests

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1 Introduction

In regular statistical experiments, the limit of the normalized likelihood ratio is always the same, because the families are LAN (see, for example, [11]). In the case of non-regular statistical models for Poisson processes, there exists a large diversity of limiting likelihood ratio processes: change-point type models lead to a log Poisson process, "cusp" type singularities provide a log fBm process, while in the models with 0-type or ∞-type singularities

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the limit processes are more sophisticated $\big($see, respectively, [14,2,4]$\big)$. Note that in change-point type models for diffusion processes, and particularly in the model of a discontinuous signal in white Gaussian noise (WGN), the limiting likelihood ratio is a log Wiener process $\big($see, for example, [11,15]$\big)$. It is interesting to investigate the relations between the different limit processes. This study was initiated in the recent works [3,7]. The present work is a part of this investigation, since we study a change-point model with variable jump size for Poissonian observations, and we obtain two different limits depending on the way the jump size is varying.

More precisely, we consider two cases. The first one corresponds to the situation when the jump size converges to a non-zero limit, while in the second one the limit is zero. The limiting likelihood ratios in these two cases are quite different. In the first case, as one could expect, the normalized likelihood ratio converges to a log Poisson process, just like the case of a fixed jump size. In the second case, the normalized likelihood ratio converges to a log Wiener process, that is, the statistical problems of parameter estimation and hypotheses testing are asymptotically equivalent to the well known problems of change-point estimation and testing for signal in WGN model. Let us note, that even if the latter result may seem unexpected, it is quite natural in the light of the recent work [3] of one of the authors, where a relation between the log Poisson and the log Wiener limiting likelihood ratios was discovered.

Let us also mention that this situation is somewhat similar to what happens in the case of multi-phase regression models, where the limiting likelihood ratio is a log compound Poisson process in the case of a fixed jump size, while it is a log Wiener process in the case of a variable jump size converging to zero $\big($see, for example, [8] and the references therein$\big)$. Note also, that the recent work [7] shades the light on the latter case, just as [3] do in our case of Poissonian observations.

Note finally, that we show not only the convergence of normalized likelihood ratios, but also the convergence of the moments of the estimators. This last convergence allows one, for example, to approximate the limiting mean square errors of the maximum likelihood and Bayesian estimators in the case of Poisson observations by the well known limiting mean square errors of these estimators calculated for signal in WGN model.

The paper is organized as follows. In Section 2 we describe the model of observations. In Section 3 we study the asymptotic behavior of the likelihood ratio. In Section 4, using the convergence of normalized likelihood ratio obtained in Section 3, we study the problem of parameter estimation. Similarly, in Section 5 we study the problem of hypothesis testing and illustrate the results by numerical simulations. Finally, Section 6 contains the proofs of all the lemmas.

2 Change-point model with variable jump size

Suppose we observe n independent realizations $X_j^{(n)} = \big\{X_j^{(n)}(t),\ t \in [0,\tau]\big\}$, $j = 1,\dots,n$, of an inhomogeneous Poisson process on the interval $[0,\tau]$ (the constant $\tau > 0$ is supposed to be known) of intensity measure

$$\Lambda_\vartheta^{(n)}(A) = \int_A \lambda_\vartheta^{(n)}(t)\,\mathrm{d}t, \qquad A \in \mathscr{B}\big([0,\tau]\big),$$

with intensity function $\lambda_\vartheta^{(n)}$, where $\vartheta \in \Theta = (\alpha,\beta)$, $0 \le \alpha < \beta \le \tau$, is some unknown parameter. The observation will be denoted $X^{(n)} = \big\{X_1^{(n)},\dots,X_n^{(n)}\big\}$ and the corresponding probability distribution will be denoted $\mathbf{P}_\vartheta^{(n)}$.

Let us note that this model of observation is equivalent to observing a single realization on the interval $[0,n\tau]$ of an inhomogeneous Poisson process with the τ-periodic intensity function coinciding with $\lambda_\vartheta^{(n)}$ on $[0,\tau]$.

The parameter ϑ corresponds to the location of a jump in the (elsewhere continuous) intensity function $\lambda_\vartheta^{(n)}$. The size of the jump (depending on n) will be denoted r_n and will be supposed converging to some $r \in \mathbb{R}$. As we will see below, the behavior of our model depends on either one has $r \neq 0$ or $r = 0$ and is quite different in these two cases.

More precisely, we assume that the following conditions are satisfied.

(C1) The intensity function $\lambda_\vartheta^{(n)}(t)$ can be written as $\lambda_\vartheta^{(n)}(t) = \psi_n(t) + r_n \mathbb{1}_{\{t>\vartheta\}}$, where the function ψ_n is continuous on $[0,\tau]$.

(C2) For all $t \in [0,\tau]$, there exist the $\lim\limits_{n\to+\infty} \psi_n(t) = \psi(t) > 0$ and, moreover, this convergence is uniform with respect to t.

(C3) As $n \to +\infty$, the jump size r_n converges to some $r \in \mathbb{R}$, that is, $r_n \to r$. In the case $r = 0$, we also suppose that this convergence ($r_n \to 0$) is slower than $n^{-1/2}$, that is, $n\,r_n^2 \to +\infty$.

(C4) The family of functions $\big\{\lambda_\vartheta^{(n)}\big\}_{n\in\mathbb{N},\vartheta\in\Theta}$ is uniformly strictly positive and uniformly bounded, that is, there exist some constants $\ell, L > 0$ such that

$$\ell \leq \lambda_\vartheta^{(n)}(t) \leq L$$

for all $n \in \mathbb{N}$, $\vartheta \in \Theta$ and $t \in [0,\tau]$.

Note that the conditions **C1** – **C3**, together with the natural condition

$$r > -\min_{t\in[0,\tau]} \psi(t), \tag{1}$$

easily imply that the condition **C4** holds for the family $\big\{\lambda_\vartheta^{(n)}\big\}_{n\geq n_0,\vartheta\in\Theta}$ with some $n_0 \in \mathbb{N}$. So, in the asymptotic setting ($n \to +\infty$), the condition **C4** can be replaced by (1), and we assume **C4** instead of the latter only for convenience (as well as in order for our model to be well defined for all $n \in \mathbb{N}$). Note also that in the case $r = 0$, the condition (1) is automatically satisfied.

An important particular case of this model is when only the jump size $\big($and not the regular part of $\lambda_\vartheta^{(n)}\big)$ depend on n. More precisely, the conditions **C1** – **C2** will be clearly met if we assume that the following condition is satisfied.

(C0) The intensity function $\lambda_\vartheta^{(n)}(t)$ can be written as $\lambda_\vartheta^{(n)}(t) = \psi(t) + r_n \mathbb{1}_{\{t>\vartheta\}}$, where the function ψ is strictly positive and continuous on $[0,\tau]$.

3 Asymptotic behavior of the likelihood ratio

The likelihood of our model is given by $\big($see, for example, [13]$\big)$

$$\begin{aligned} L_n\big(\vartheta, X^{(n)}\big) &= \exp\bigg\{\sum_{j=1}^n \int_{[0,\tau]} \ln \lambda_\vartheta^{(n)}(t)\, X_j^{(n)}(\mathrm{d}t) - n\int_0^\tau \big[\lambda_\vartheta^{(n)}(t) - 1\big]\,\mathrm{d}t\bigg\} \\ &= \exp\bigg\{\sum_{j=1}^n \sum_{i\in I_j^{(n)}} \ln \lambda_\vartheta^{(n)}(t_{j,i}) - n\int_0^\tau \big[\lambda_\vartheta^{(n)}(t) - 1\big]\,\mathrm{d}t\bigg\}, \end{aligned} \tag{2}$$

where $t_{j,i}$, $i \in I_j^{(n)}$, are the jump times of the process $X_j^{(n)}$. Note that as function of ϑ, each $\lambda_\vartheta^{(n)}(t_{j,i})$ is discontinuous (has a jump and is right continuous) at $\vartheta = t_{j,i}$. So, $L_n\bigl(\,\cdot\,, X^{(n)}\bigr)$ is a random process with càdlàg (continuous from the right and having finite limits from the left) trajectories.

We put $\varphi_n = \frac{1}{n}$ in the case $r \neq 0$ and $\varphi_n = \frac{1}{n r_n^2}$ in the case $r = 0$, and we introduce the normalized likelihood ratio

$$\begin{aligned} Z_{n,\vartheta}(u) &= \frac{L_n\bigl(\vartheta + u\varphi_n, X^{(n)}\bigr)}{L_n\bigl(\vartheta, X^{(n)}\bigr)} \\ &= \exp\biggl\{\sum_{j=1}^n \int_{[0,\tau]} \ln \frac{\lambda_{\vartheta+u\varphi_n}^{(n)}(t)}{\lambda_\vartheta^{(n)}(t)}\, X_j^{(n)}(\mathrm{d}t) - n\int_0^\tau \bigl(\lambda_{\vartheta+u\varphi_n}^{(n)}(t) - \lambda_\vartheta^{(n)}(t)\bigr)\,\mathrm{d}t\biggr\} \\ &= \exp\biggl\{\sum_{j=1}^n \sum_{i\in I_j^{(n)}} \ln \frac{\lambda_{\vartheta+u\varphi_n}^{(n)}(t_{j,i})}{\lambda_\vartheta^{(n)}(t_{j,i})} - n\int_0^\tau \bigl(\lambda_{\vartheta+u\varphi_n}^{(n)}(t) - \lambda_\vartheta^{(n)}(t)\bigr)\,\mathrm{d}t\biggr\}, \end{aligned}$$

where $u \in U_n = \bigl(\varphi_n^{-1}(\alpha - \vartheta), \varphi_n^{-1}(\beta - \vartheta)\bigr)$.

Note that in both cases we have (by the condition **C3** in the case $r = 0$) $\varphi_n \to 0$.

Note also that if $u > 0$, we can rewrite $Z_{n,\vartheta}(u)$ as

$$\begin{aligned} Z_{n,\vartheta}(u) &= \exp\biggl\{\sum_{j=1}^n \int_{(\vartheta,\vartheta+u\varphi_n]} \ln \frac{\psi_n(t)}{\psi_n(t) + r_n}\, X_j^{(n)}(\mathrm{d}t) + n\int_\vartheta^{\vartheta+u\varphi_n} r_n\,\mathrm{d}t\biggr\} \\ &= \exp\biggl\{\sum_{j=1}^n \sum_i \ln \frac{\psi_n(t_{j,i})}{\psi_n(t_{j,i}) + r_n} + u\, r_n^\gamma\biggr\}. \end{aligned} \tag{3}$$

In the last expression the inner sum is taken over the set $\bigl\{i \in I_j^{(n)} \,:\, \vartheta < t_{j,i} \leq \vartheta + u\varphi_n\bigr\}$ and we have $\gamma = 1$ in the case $r \neq 0$ and $\gamma = -1$ in the case $r = 0$.

Similarly, if $u < 0$, we have

$$\begin{aligned} Z_{n,\vartheta}(u) &= \exp\biggl\{\sum_{j=1}^n \int_{(\vartheta+u\varphi_n,\vartheta]} \ln \frac{\psi_n(t) + r_n}{\psi_n(t)}\, X_j^{(n)}(\mathrm{d}t) - n\int_{\vartheta+u\varphi_n}^{\vartheta} r_n\,\mathrm{d}t\biggr\} \\ &= \exp\biggl\{\sum_{j=1}^n \sum_i \ln \frac{\psi_n(t_{j,i}) + r_n}{\psi_n(t_{j,i})} + u\, r_n^\gamma\biggr\}, \end{aligned}$$

where the last sum is taken over the set $\bigl\{i \in I_j^{(n)} \,:\, \vartheta + u\varphi_n < t_{j,i} \leq \vartheta\bigr\}$ and γ is as above.

Note equally, that the process $\ln Z_{n,\vartheta}$ has independent increments. Indeed, its increments on disjoint intervals involve stochastic integrals (of a deterministic function with respect to Poisson processes) on disjoint intervals, and hence are independent. In other words, using the terminology of Strasser [18], our model is an "experiment with independent increments". Note also, that in this case the process $Z_{n,\vartheta}$ $\bigl($as well as, for example, the process $Z_{n,\vartheta}^{1/2}\bigr)$ is clearly a Markov process.

Note finally, that the trajectories of the process $Z_{n,\vartheta}$ are càdlàg functions. Moreover, correctly extending these trajectories to the whole real line, one can consider that they belong to the Skorohod space $\mathscr{D}_0(\mathbb{R})$. This space is defined as the space of functions f on $\mathbb{R}$ which do not have discontinuities of the second kind and which are vanishing at infinity, that is, such that $\lim\limits_{u\to\pm\infty} f(u) = 0$. We assume that all the functions $f \in \mathscr{D}_0(\mathbb{R})$ are continuous from the right (are càdlàg).

Let us recall that the Skorohod metric on the space $\mathscr{D}_0(\mathbb{R})$ is introduced by

$$d(f,g) = \inf_{\lambda}\left[\sup_{u\in\mathbb{R}}\bigl|f(u)-g\bigl(\lambda(u)\bigr)\bigr| + \sup_{u\in\mathbb{R}}|u-\lambda(u)|\right],$$

where the inf is taken over all strictly increasing continuous one-to-one mappings $\lambda:\mathbb{R}\to\mathbb{R}$.

Let us also recall a criterion of weak convergence in $\mathscr{D}_0(\mathbb{R})$. We put

$$\Delta_h(f) = \sup_{u\in\mathbb{R}}\sup_{u',u''}\Bigl[\min\bigl\{\bigl|f(u')-f(u)\bigr|,\bigl|f(u'')-f(u)\bigr|\bigr\}\Bigr] + \sup_{|u|>1/h}|f(u)|,$$

where the inner sup is over all u',u'' such that $u-h\leq u'<u\leq u''<u+h$. A criterion of weak convergence in $\mathscr{D}_0(\mathbb{R})$ is given in the following lemma (see [9] for more details).

Lemma 1 *Let $z_{n,\vartheta}$, $n\in\mathbb{N}$, and z_ϑ be random processes with realizations belonging to $\mathscr{D}_0(\mathbb{R})$ with probability 1. If, as $n\to+\infty$, the finite dimensional distributions of $z_{n,\vartheta}$ converge uniformly in $\vartheta\in\mathbb{K}$ to the finite dimensional distributions of z_ϑ, and if for any $\delta>0$*

$$\lim_{h\to 0}\sup_{n\in\mathbb{N},\vartheta\in\mathbb{K}}\mathbf{P}\bigl\{\Delta_h(z_{n,\vartheta})>\delta\bigr\}=0, \tag{4}$$

then, uniformly in $\vartheta\in\mathbb{K}$, the process $z_{n,\vartheta}$ converges weakly in the space $\mathscr{D}_0(\mathbb{R})$ to the process z_ϑ.

Note that here and in the sequel $\mathbb{K}$ denotes an arbitrary compact in Θ.

The main objective of this section is the study of the asymptotic behavior (in the sense of the weak convergence in the space $\mathscr{D}_0(\mathbb{R})$ as $n\to\infty$) of the above introduced normalized likelihood ratio $Z_{n,\vartheta}$. This behavior depends on either one has $r\neq 0$ or $r=0$ and is quite different in these two cases, so the limit process must be introduced in a different manner in these two cases.

Case $r\neq 0$ limit process In the case $r\neq 0$, the limit process is a log Poisson type process and is introduced by

$$Z_\vartheta(u) = \begin{cases}\exp\Bigl\{\ln\frac{\psi(\vartheta)}{\psi(\vartheta)+r}X^+(u)+ru\Bigr\}, & \text{if } u\geq 0,\\ \exp\Bigl\{\ln\frac{\psi(\vartheta)+r}{\psi(\vartheta)}X^-\bigl((-u)-\bigr)+ru\Bigr\}, & \text{if } u<0,\end{cases}$$

where X^+ and X^- are independent Poisson processes on $\mathbb{R}_+$ of constant intensities $\psi(\vartheta)+r$ and $\psi(\vartheta)$ respectively.

Let us note that $Z_\vartheta(u)\overset{\mathrm{d}}{=}Z^*_\rho(-ru)$ with the constant $\rho=\bigl|\ln\frac{\psi(\vartheta)}{\psi(\vartheta)+r}\bigr|$ and the process Z^*_ρ defined by

$$Z^*_\rho(v) = \begin{cases}\exp\bigl\{\rho Y^+(v)-v\bigr\}, & \text{if } v\geq 0,\\ \exp\bigl\{-\rho Y^-\bigl((-v)-\bigr)-v\bigr\}, & \text{if } v<0,\end{cases}$$

where Y^+ and Y^- are independent Poisson processes on $\mathbb{R}_+$ of constant intensities $\frac{1}{e^\rho-1}$ and $\frac{1}{1-e^{-\rho}}$ respectively.

Note also that the process Z^*_ρ was recently studied in [3] and that its trajectories (as well as those of the process Z_ϑ) almost surely belong to the space $\mathscr{D}_0(\mathbb{R})$. (More rigorously, in order to keep all the trajectories in the space $\mathscr{D}_0(\mathbb{R})$, above we should rather have written $Z_\vartheta(u)\overset{\mathrm{d}}{=}Z^*_\rho\bigl((-ru)-\bigr)$ in the case $r>0$).

Case $r = 0$ limit process In the case $r = 0$, the limit process is a log Wiener type process and is introduced by

$$Z_\vartheta(u) = \exp\left\{\psi^{-1/2}(\vartheta)\,W(u) - \frac{|u|}{2\psi(\vartheta)}\right\}, \quad u \in \mathbb{R},$$

where $W(u)$, $u \in \mathbb{R}$, is a double-sided Brownian motion (Wiener process).

Let us note that $Z_\vartheta(u) \stackrel{\mathrm{d}}{=} Z^*\big(u/\psi(\vartheta)\big)$ with the process Z^* defined by

$$Z^*(v) = \exp\left\{W(v) - \frac{|v|}{2}\right\}, \quad v \in \mathbb{R}. \tag{5}$$

Note also that the trajectories of the processes Z^* and Z_ϑ almost surely belong to the space $\mathscr{C}_0(\mathbb{R})$ of continuous functions on $\mathbb{R}$ vanishing at infinity, and that $\mathscr{C}_0(\mathbb{R}) \subset \mathscr{D}_0(\mathbb{R})$.

Now we can state the following theorem about the asymptotic behavior of the normalized likelihood ratio.

Theorem 1 *Let the conditions* **C1 – C4** *be fulfilled. Then, uniformly in $\vartheta \in \mathbb{K}$, the process $Z_{n,\vartheta}$ converges weakly in the space $\mathscr{D}_0(\mathbb{R})$ to the process Z_ϑ.*

Let us also remark, that sometimes it may be more convenient to use a slightly different rate for introducing the normalized likelihood ratio. More precisely, one can use the rate $\varphi_n^* = \frac{1}{|r|n}$ (rather than $\varphi_n = \frac{1}{n}$) in the case $r \neq 0$, and the rate $\varphi_n^* = \frac{\psi(\vartheta)}{n r_n^2}$ (rather than $\varphi_n = \frac{1}{n r_n^2}$) in the case $r = 0$. That is, one can consider (instead of $Z_{n,\vartheta}$) the normalized likelihood ratio $Z_{n,\vartheta}^*$ defined by

$$Z_{n,\vartheta}^*(v) = \frac{L_n\big(\vartheta + v\,\varphi_n^*, X^{(n)}\big)}{L_n\big(\vartheta, X^{(n)}\big)} = Z_{n,\vartheta}\big(c\,v\big)$$

with $c = 1/|r|$ in the case $r \neq 0$, and $c = \psi(\theta)$ in the case $r = 0$. Then, Theorem 1 will be clearly transformed to the following (equivalent) statement.

Theorem 2 *Let the conditions* **C1 – C4** *be fulfilled. Then, uniformly in $\vartheta \in \mathbb{K}$, the process $Z_{n,\vartheta}^*$ converges weakly in the space $\mathscr{D}_0(\mathbb{R})$ to*

- *the process Z_ρ^*, in the case $r < 0$,*
- *the process $Z_\rho^\star$ defined by $Z_\rho^\star(v) = Z_\rho^*\big((-v)-\big)$, in the case $r > 0$,*
- *the process Z^*, in the case $r = 0$.*

The proof of Theorem 1 consist in checking the criterion of week convergence given in Lemma 1. For this, we follow the methods and ideas used in [11, Chapters 5.3 and 5.4] and establish several lemmas (the proofs of the lemmas are in Section 6).

Lemma 2 *Let the conditions* **C1 – C4** *be fulfilled. Then the finite-dimensional distributions of the process $Z_{n,\vartheta}$ converge to those of the process Z_ϑ, and this convergence is uniform with respect to $\vartheta \in \mathbb{K}$.*

Lemma 3 *Let the conditions* **C1 – C4** *be fulfilled. Then there exists a constant $C > 0$ such that*

$$\mathbf{E}_\vartheta^{(n)}\big|Z_{n,\vartheta}^{1/2}(u_1) - Z_{n,\vartheta}^{1/2}(u_2)\big|^2 \leq C\,|u_1 - u_2|$$

for all $n \in \mathbb{N}$, $u_1, u_2 \in U_n$ and $\vartheta \in \mathbb{K}$.

Lemma 4 *Let the conditions* **C1 – C4** *be fulfilled. Then there exists a constant $k_* > 0$ such that*

$$\mathbf{E}_\vartheta^{(n)} Z_{n,\vartheta}^{1/2}(u) \leq \exp\big\{-k_*\,|u|\big\}$$

for all $u \in U_n$, $\vartheta \in \mathbb{K}$ and sufficiently large values of n (all $n \in \mathbb{N}$ in the case $r = 0$).

Final argument of the proof of Theorem 1 in the case $r \neq 0$ In this case, defining $Z^{1/2}_{n,\vartheta;\mathrm{a.c.}}$ to be the absolutely continuous component of the function $Z^{1/2}_{n,\vartheta}$ and, for $p = 1,2$, denoting $A_p = A_p(u,u+h)$ the event that $Z_{n,\vartheta}$ has at least p jumps on the interval $(u,u+h)$, we also have the following lemma.

Lemma 5 *Let the conditions* **C1 – C4** *be fulfilled with $r \neq 0$. Then the inequalities*

$$\mathbf{E}^{(n)}_{\vartheta} \big|Z^{1/2}_{n,\vartheta;\mathrm{a.c.}}(u+h) - Z^{1/2}_{n,\vartheta;\mathrm{a.c.}}(u)\big|^2 \leq Ch^2,$$
$$\mathbf{P}^{(n)}_{\vartheta}(A_1) \leq D_1 h \tag{6}$$

and

$$\mathbf{P}^{(n)}_{\vartheta}(A_2) \leq D_2 h^2 \tag{7}$$

hold with certain constants $C, D_1, D_2 > 0$ (independent of n, ϑ, u and h).

Now, with the help of the above lemmas, we can finish the proof of Theorem 1 in the case $r \neq 0$ following the standard argument of [11, Chapters 5.3 and 5.4]. More precisely, the weak convergence in $\mathscr{D}_0(\mathbb{R})$ of the processes $Z_{n,\vartheta}$ to the process Z_ϑ follows from Theorem 5.4.2 of [13], which is, in fact, contained in [11] (without being formulated there). Note, that the conditions of this theorem are nothing but Lemmas 2, 4 and 5, and that its proof consist in verifying the condition (4).

Final argument of the proof of Theorem 1 in the case $r = 0$ In this case, it is not possible to establish a lemma similar to Lemma 5. In particular, the inequalities (6) and (7) do not hold, since in this case (in contrary to the case $r = 0$) the jumps are not becoming seldom. More precisely, as $n \to +\infty$, instead of having (on any finite interval) few “non-vanishing” jumps, one has more and more jumps which at the same time become smaller and smaller (which explains that the trajectories of the limiting likelihood ratio process in this case are continuous but nowhere differentiable functions). So, in order to finish the proof of Theorem 1 we use a different technique.

Since the increments of the process $\ln Z_{n,\vartheta}$ are independent, the convergence of its restrictions (and hence of those of $Z_{n,\vartheta}$) on finite intervals $[A,B] \subset \mathbb{R}$ $\big($that is, convergence in the Skorohod space $\mathscr{D}\big([A,B]\big)$ of functions on $[A,B]$ without discontinuities of the second kind$\big)$ follows from Theorem 6.5.5 of Gihman and Skorohod [10], Lemma 2 and the following lemma.

Lemma 6 *Let the conditions* **C1 – C4** *be fulfilled with $r = 0$. Then for any $\varepsilon > 0$ we have*

$$\lim_{h\to 0} \lim_{n\to+\infty} \sup_{|u_1-u_2|<h} \mathbf{P}^{(n)}_{\vartheta}\big(\big|\ln Z_{n,\vartheta}(u_1) - \ln Z_{n,\vartheta}(u_2)\big| > \varepsilon\big) = 0.$$

for all $u_1, u_2 \in U_n$ and $\vartheta \in \mathbb{K}$.

Let us note, that taking a closer look on the proof of this lemma, one can see that we have even a stronger result: for any $\varepsilon > 0$ we have

$$\lim_{h\to 0} \lim_{n\to+\infty} \sup_{\vartheta\in\mathbb{K}} \sup_{|u_1-u_2|<h} \mathbf{P}^{(n)}_{\vartheta}\big(\big|\ln Z_{n,\vartheta}(u_1) - \ln Z_{n,\vartheta}(u_2)\big| > \varepsilon\big) = 0$$

for all $u_1, u_2 \in U_n$, which allow us to conclude that the convergence of the restrictions of the process $Z_{n,\vartheta}$ on finite intervals $[A,B] \subset \mathbb{R}$ to those of the process Z_ϑ is uniform with respect to $\vartheta \in \mathbb{K}$. Note also, that an alternative way to prove this convergence (instead of using Lemmas 2 and 6) is to study the characteristics of the processes and apply, for example, Theorem 7.3.4 of Jacod and Shiryaev [12]. However, in our opinion, the proof given here gives more insight on the structure of the considered processes.

In order to conclude the proof of Theorem 1 applying the criterion of week convergence in $\mathscr{D}_0(\mathbb{R})$ given in Lemma 1, we need to check the condition (4). Since we have already established the convergence of the restrictions on finite intervals $[A,B] \subset \mathbb{R}$, it remains to control the second term of the modulus of continuity $\Delta_h(Z_{n,\vartheta})$ $\big($see, for example, [11, Chapters 5.3 and 5.4]$\big)$. So, the last ingredient of the proof of Theorem 1 is the following estimate on the tails of the process $Z_{n,\vartheta}$.

Lemma 7 *Let the conditions* **C1 – C4** *be fulfilled with* $r = 0$*. Then there exist some constants* $b, C > 0$ *such that*

$$\mathbf{P}_\vartheta^{(n)}\left(\sup_{|u|>D} Z_{n,\vartheta}(u) > e^{-bD}\right) \leq Ce^{-bD} \tag{8}$$

for all $D \geq 0$, $n \in \mathbb{N}$ *and* $\vartheta \in \mathbb{K}$.

4 Parameter estimation

In this section we apply the convergence of normalized likelihood ratio obtained in Section 3 to study the problem of parameter estimation for our model of observations. In the case $r \neq 0$, the limiting likelihood ratio being the same as in the fixed jump size case, the properties of estimators are also the same $\big($see, for example, [13, 14] for more details$\big)$. So, here we consider the case $r = 0$ only.

Recall that as function of ϑ, the likelihood of our model given by (2) is discontinuous (has jumps). So, the maximum likelihood estimator $\widehat{\vartheta}_n$ of ϑ is introduced through the equation

$$\max\Big\{L_n\big(\widehat{\vartheta}_n+, X^{(n)}\big), L_n\big(\widehat{\vartheta}_n-, X^{(n)}\big)\Big\} = \sup_{\vartheta\in\Theta} L_n\big(\vartheta, X^{(n)}\big).$$

The Bayesian estimator $\widetilde{\vartheta}_n$ of ϑ for a given prior density p and for square loss is defined by

$$\widetilde{\vartheta}_n = \frac{\int_\alpha^\beta \vartheta\, p(\vartheta) L_n\big(\vartheta, X^{(n)}\big)\,\mathrm{d}\vartheta}{\int_\alpha^\beta p(\vartheta) L_n\big(\vartheta, X^{(n)}\big)\,\mathrm{d}\vartheta}.$$

We are interested in the asymptotic properties of the maximum likelihood and Bayesian estimators of ϑ as $n \to +\infty$. To describe the properties of the estimators we need some additional notations.

We introduce the random variables ξ_ϑ, ξ^*, ζ_ϑ and ζ^* by the equations

$$Z_\vartheta(\xi_\vartheta) = \sup_{u\in\mathbb{R}} Z_\vartheta(u),$$
$$Z^*(\xi^*) = \sup_{u\in\mathbb{R}} Z^*(u),$$

$$\zeta_\vartheta = \frac{\int_{-\infty}^{+\infty} u\, Z_\vartheta(u)\,\mathrm{d}u}{\int_{-\infty}^{+\infty} Z_\vartheta(u)\,\mathrm{d}u}$$

and

$$\zeta^* = \frac{\int_{-\infty}^{+\infty} u\, Z^*(u)\,\mathrm{d}u}{\int_{-\infty}^{+\infty} Z^*(u)\,\mathrm{d}u}\,.$$

Let us note that $\xi_\vartheta \stackrel{\mathrm{d}}{=} \psi(\vartheta)\,\xi^*$ and $\zeta_\vartheta \stackrel{\mathrm{d}}{=} \psi(\vartheta)\,\zeta^*$.

Now we can state the following theorem giving an asymptotic lower bound on the risk of all the estimators of ϑ.

Theorem 3 *Let the conditions* **C1 – C4** *be fulfilled with $r = 0$. Then, for any $\vartheta_0 \in \Theta$, we have*

$$\lim_{\delta\to 0}\ \varliminf_{n\to+\infty}\ \inf_{\overline{\vartheta}_n}\ \sup_{|\vartheta-\vartheta_0|<\delta}\ \varphi_n^{-2}\,\mathbf{E}_\vartheta^{(n)}(\overline{\vartheta}_n-\vartheta)^2 \geq \mathbf{E}\zeta_{\vartheta_0}^2 = \psi^2(\vartheta_0)\,\mathbf{E}(\zeta^*)^2,$$

where the inf *is taken over all possible estimators $\overline{\vartheta}_n$ of the parameter ϑ.*

This theorem allows us to introduce the following definition.

Definition 1 Let the conditions ***C1*** – ***C4*** be fulfilled with $r = 0$. We say that an estimator ϑ_n^* is asymptotically efficient if

$$\lim_{\delta\to 0}\ \lim_{n\to+\infty}\ \sup_{|\vartheta-\vartheta_0|<\delta}\ \varphi_n^{-2}\,\mathbf{E}_\vartheta^{(n)}(\vartheta_n^*-\vartheta)^2 = \mathbf{E}\zeta_{\vartheta_0}^2 = \psi^2(\vartheta_0)\,\mathbf{E}(\zeta^*)^2$$

for all $\vartheta_0 \in \Theta$.

Now, we can state the following two theorems giving the asymptotic properties of the maximum likelihood and Bayesian estimators.

Theorem 4 *Let the conditions* **C1 – C4** *be fulfilled with $r = 0$. Then the maximum likelihood estimator $\widehat{\vartheta}_n$ satisfies uniformly on $\vartheta \in \mathbb{K}$ the relations*

$$\mathbf{P}_\vartheta^{(n)} - \lim_{n\to+\infty} \widehat{\vartheta}_n = \vartheta,$$

$$\mathscr{L}_\vartheta^{(n)}\big\{\varphi_n^{-1}(\widehat{\vartheta}_n-\vartheta)\big\} \Rightarrow \mathscr{L}(\xi_\vartheta) = \mathscr{L}\big(\psi(\vartheta)\xi^*\big)$$

and

$$\lim_{n\to+\infty} \mathbf{E}_\vartheta^{(n)}\varphi_n^{-p}\big|\widehat{\vartheta}_n-\vartheta\big|^p = \mathbf{E}\,|\xi_\vartheta|^p = \psi^p(\vartheta)\,\mathbf{E}\,|\xi^*|^p \quad \textit{for any} \quad p>0.$$

In particular, the relative asymptotic efficiency of $\widehat{\vartheta}_n$ is $\mathbf{E}(\zeta^*)^2/\mathbf{E}(\xi^*)^2$.

Theorem 5 *Let the conditions* **C1 – C4** *be fulfilled with $r = 0$. Then, for any continuous strictly positive density, the Bayesian estimator $\widetilde{\vartheta}_n$ satisfies uniformly on $\vartheta \in \mathbb{K}$ the relations*

$$\mathbf{P}_\vartheta^{(n)} - \lim_{n\to+\infty} \widetilde{\vartheta}_n = \vartheta,$$

$$\mathscr{L}_\vartheta^{(n)}\big\{\varphi_n^{-1}(\widetilde{\vartheta}_n-\vartheta)\big\} \Rightarrow \mathscr{L}(\zeta_\vartheta) = \mathscr{L}\big(\psi(\vartheta)\zeta^*\big)$$

and

$$\lim_{n\to+\infty} \mathbf{E}_\vartheta^{(n)}\varphi_n^{-p}\big|\widetilde{\vartheta}_n-\vartheta\big|^p = \mathbf{E}\,|\zeta_\vartheta|^p = \psi^p(\vartheta)\,\mathbf{E}\,|\zeta^*|^p \quad \textit{for any} \quad p>0.$$

In particular, $\widetilde{\vartheta}_n$ is asymptotically efficient.

Theorems 3–5 follow from the properties of the normalized likelihood ratio established in Section 3. More precisely, Theorem 5 is a consequence of Lemmas 2–4 and [11, Theorem 1.10.2]. Having the properties of the Bayesian estimators given in Theorem 5, we can cite [11, Theorem 1.9.1] to provide the proof of Theorem 3. Finally, the proof of Theorem 4 can be carried out following the standard argument of [11, Chapters 5.3 and 5.4] which is based on the weak convergence established in Theorem 1 together with the inequality (8).

5 Hypothesis testing

In this section we apply the convergence of normalized likelihood ratio obtained in Section 3 to study the problem of hypothesis testing for our model of observations. In the case $r \neq 0$, the limiting likelihood ratio being the same as in the fixed jump size case, the properties of test are also the same $\bigl($see [6] for more details$\bigr)$. So, here we consider the case $r = 0$ only.

We consider the same model of observation as above, with the only difference that now we suppose that $\theta \in \Theta = [\vartheta_1, b)$, $0 < \vartheta_1 < \beta \leq \tau$. We assume that the conditions **(C1)–(C4)** are fulfilled with $r = 0$ and we want to test the following two hypothesis:

$$\begin{aligned} \mathscr{H}_1 &\quad : \quad \vartheta = \vartheta_1, \\ \mathscr{H}_2 &\quad : \quad \vartheta > \vartheta_1. \end{aligned}$$

We define a (randomized) test $\overline{\phi}_n = \overline{\phi}_n\bigl(X^{(n)}\bigr)$ as the probability to accept the hypothesis $\mathscr{H}_2$. The size of the test is defined by $\mathbf{E}_{\vartheta_1}^{(n)} \overline{\phi}_n\bigl(X^{(n)}\bigr)$, and its power function is given by $\beta(\overline{\phi}_n, \vartheta) = \mathbf{E}_{\vartheta}^{(n)} \overline{\phi}_n\bigl(X^{(n)}\bigr)$, $\vartheta > \vartheta_1$. As usually, we denote $\mathscr{K}_\varepsilon$ the class of tests of asymptotic size $\varepsilon \in [0,1]$, that is,

$$\mathscr{K}_\varepsilon = \Bigl\{ \overline{\phi}_n \quad : \quad \lim_{n \to +\infty} \mathbf{E}_{\vartheta_1}^{(n)} \overline{\phi}_n\bigl(X^{(n)}\bigr) = \varepsilon \Bigr\}.$$

Our goal is to construct some tests belonging to this class and to compare them. The comparison of tests can be done by comparison of their power functions. It is known that for any reasonable test and for any fixed alternative the power function tends to 1. To avoid this difficulty, we use Pitman's approach and consider *contiguous* (or *close*) alternatives. More precisely, changing the variable $\vartheta = \vartheta_u \stackrel{\Delta}{=} \vartheta_1 + u\varphi_n^*$, where $\varphi_n^* = \frac{\psi(\vartheta_1)}{n r_n^2}$, the initial problem of hypotheses testing can be replaced by the following one

$$\begin{aligned} \mathscr{H}_1 &\quad : \quad u = 0, \\ \mathscr{H}_2 &\quad : \quad u > 0, \end{aligned}$$

and the power function is now $\beta(\overline{\phi}_n, u) = \mathbf{E}_{\vartheta_n}^{(n)} \overline{\phi}_n\bigl(X^{(n)}\bigr)$, $u > 0$.

The study is essentially based on the properties of the normalized likelihood ratio established above. Note that the limit of the normalized likelihood ratio at the point $\vartheta = \vartheta_1$ (under hypothesis $\mathscr{H}_1$) is the following:

$$Z_{n,\vartheta_1}^*(v) = \frac{L_n\bigl(\vartheta_1 + v\varphi_n^*, X^{(n)}\bigr)}{L_n\bigl(\vartheta_1, X^{(n)}\bigr)} \Rightarrow Z^*(v), \quad v \geq 0,$$

where the process Z^* is defined by (5).

Under alternatives, we obtain

$$\begin{aligned}
Z^*_{n,\vartheta_1}(v) &= \frac{L_n\left(\vartheta_1+v\varphi_n^*,X^{(n)}\right)}{L_n\left(\vartheta_1,X^{(n)}\right)}\\
&= \left(\frac{L_n\left(\vartheta_1,X^{(n)}\right)}{L_n\left(\vartheta_u,X^{(n)}\right)}\right)^{-1} \frac{L_n\left(\vartheta_1+v\varphi_n^*,X^{(n)}\right)}{L_n\left(\vartheta_u,X^{(n)}\right)}\\
&= \left(\frac{L_n\left(\vartheta_u-u\varphi_n^*,X^{(n)}\right)}{L_n\left(\vartheta_u,X^{(n)}\right)}\right)^{-1} \frac{L_n\left(\vartheta_u+(v-u)\varphi_n^*,X^{(n)}\right)}{L_n\left(\vartheta_u,X^{(n)}\right)}\\
&\Rightarrow \left(Z^*(-u)\right)^{-1} Z^*(v-u) \overset{\mathrm{d}}{=} \exp\left\{W(v)-\frac{|v-u|}{2}+\frac{u}{2}\right\} \overset{\Delta}{=} Z^*_u(v).
\end{aligned}$$

The score-function test — which is locally asymptotically uniformly most powerful (LAUMP) in the regular case $\big($see [5]$\big)$— does not exist in this non-regular situation. So, we will construct and study the general likelihood ratio test (GLRT), Wald's test (WT) and two Bayesian tests (BT1 and BT2).

General likelihood ratio test The GLRT is defined by the relations

$$\widehat{\phi}_n\left(X^{(n)}\right) = \mathbb{1}_{\left\{Q(X^{(n)})>h_\varepsilon\right\}},$$

with

$$Q\left(X^{(n)}\right) = \sup_{\vartheta>\vartheta_1} \frac{L_n\left(\vartheta,X^{(n)}\right)}{L_n\left(\vartheta_1,X^{(n)}\right)} = \max\left\{\frac{L_n\left(\widehat{\vartheta}_n+,X^{(n)}\right)}{L_n\left(\vartheta_1,X^{(n)}\right)}\ ,\ \frac{L_n\left(\widehat{\vartheta}_n-,X^{(n)}\right)}{L_n\left(\vartheta_1,X^{(n)}\right)}\right\},$$

where $\widehat{\vartheta}_n$ is the maximum likelihood estimator of θ.

To choose the threshold h_ε such that $\widehat{\phi}_n\left(X^{(n)}\right) \in \mathscr{K}_\varepsilon$ we need to solve the following equation (under hypothesis $\mathscr{H}_1$)

$$\mathbf{P}^{(n)}_{\vartheta_1}\left\{Q\left(X^{(n)}\right) > h_\varepsilon\right\} = \mathbf{P}^{(n)}_{\vartheta_1}\left\{\sup_{v>0} Z^*_{n,\vartheta_1}(v) > h_\varepsilon\right\} \to \mathbf{P}\left\{\sup_{v>0} Z^*(v) > h_\varepsilon\right\} = \varepsilon.$$

For this, we note that the random variable $\sup_{v>0} \ln Z^*(v)$ has the exponential distribution with parameter 1 $\big($see, for example, [1]$\big)$. This allows us to calculate explicitly the threshold h_ε of the GLRT as solution of the equation $1-e^{-\ln h_\varepsilon} = 1-\varepsilon$, that is, $h_\varepsilon = 1/\varepsilon$.

The power function of the GLRT has the following limit:

$$\beta\left(\widehat{\phi}_n,u\right) = \mathbf{P}^{(n)}_{\vartheta_u}\left\{\sup_{v>0} Z^*_{n,\vartheta_1}(v) > h_\varepsilon\right\} \to \mathbf{P}\left\{\sup_{v>0} Z^*_u(v) > h_\varepsilon\right\}.$$

This limiting power function is obtained below with the help of numerical simulations.

Wald's test To define the WT, let us note that the maximum likelihood estimator $\widehat{\vartheta}_n$ converges in distribution:

$$(\varphi_n^*)^{-1}\left(\widehat{\vartheta}_n-\vartheta_1\right)\Rightarrow \xi_+^*,$$

where the random variable ξ_+^* is solution of the equation

$$Z^*(\xi_+^*)=\sup_{v>0} Z^*(v).$$

Therefore, if we put

$$\phi_n^\circ\left(X^{(n)}\right)=\mathbb{1}_{\left\{(\varphi_n^*)^{-1}\left(\widehat{\vartheta}_n-\vartheta_1\right)>m_\varepsilon\right\}},$$

where m_ε is defined by the equation

$$\mathbf{P}\left\{\xi_+^*>m_\varepsilon\right\}=\varepsilon,$$

then $\phi_n^\circ\in\mathscr{K}_\varepsilon$.

We recall the result of [17], that the joint distribution of $\left(\ln Z^*\left(\xi_+^*\right),\xi_+^*\right)$ has the density

$$f(y,t)=\frac{y}{\sqrt{2\pi t^3}}\exp\left\{-\frac{\left(y+\frac{t}{2}\right)^2}{2t}\right\},$$

which allows us to calculate the marginal density of ξ_+^* as follows:

$$\begin{aligned}
f(t)&=\int_0^{+\infty} f(y,t)\,\mathrm{d}y=\int_0^{+\infty}\frac{\frac{y}{\sqrt{t}}}{\sqrt{2\pi t}}\exp\left\{-\frac{1}{2}\left(\frac{y}{\sqrt{t}}+\frac{\sqrt{t}}{2}\right)^2\right\}\mathrm{d}\left(\frac{y}{\sqrt{t}}\right)\\
&=\int_0^{+\infty}\frac{z}{\sqrt{2\pi t}}\exp\left\{-\frac{1}{2}\left(z+\frac{\sqrt{t}}{2}\right)^2\right\}\mathrm{d}z=\int_{\frac{\sqrt{t}}{2}}^{+\infty}\frac{x-\frac{\sqrt{t}}{2}}{\sqrt{2\pi t}}\exp\left\{-\frac{x^2}{2}\right\}\mathrm{d}x\\
&=-\int_{\frac{\sqrt{t}}{2}}^{+\infty}\frac{1}{\sqrt{2\pi t}}\,\mathrm{d}\exp\left\{-\frac{x^2}{2}\right\}-\int_{\frac{\sqrt{t}}{2}}^{+\infty}\frac{\frac{\sqrt{t}}{2}}{\sqrt{2\pi t}}\exp\left\{-\frac{x^2}{2}\right\}\mathrm{d}x\\
&=\frac{1}{\sqrt{2\pi t}}\exp\left\{-\frac{t}{8}\right\}-\frac{1}{2}\Phi\left(-\frac{\sqrt{t}}{2}\right),
\end{aligned}$$

where Φ is the distribution function of the standard Gaussian low $\mathcal{N}(0,1)$. So, the threshold m_ε can be obtained as the solution of the equation

$$\int_{m_\varepsilon}^{+\infty}\left(\frac{1}{\sqrt{2\pi t}}\exp\left\{-\frac{t}{8}\right\}-\frac{1}{2}\Phi\left(-\frac{\sqrt{t}}{2}\right)\right)\mathrm{d}t=\varepsilon. \tag{9}$$

The power function of the WT has the following limit:

$$\beta\left(\phi_n^\circ,u\right)=\mathbf{P}_{\vartheta_u}^{(n)}\left\{(\varphi_n^*)^{-1}\left(\widehat{\vartheta}_n-\vartheta_u\right)+u>m_\varepsilon\right\}\to\mathbf{P}\left\{\xi_u^*>m_\varepsilon-u\right\},$$

where the random variable ξ_u^* is solution of the equation

$$Z\left(\xi_u^*\right)=\sup_{v>-u} Z^*(v).$$

Note that we can also derive another expression of the limiting power function of the WT as follows:

$$\beta\left(\phi_n^\circ,u\right)=\mathbf{P}_{\vartheta_u}^{(n)}\left\{(\varphi_n^*)^{-1}\left(\widehat{\vartheta}_n-\vartheta_1\right)>m_\varepsilon\right\}\to\mathbf{P}\left\{\xi_{u,+}^*>m_\varepsilon\right\},$$

where the random variable $\xi_{u,+}^*$ is solution of the equation

$$Z\left(\xi_{u,+}^*\right) = \sup_{v>0} Z_u^*\left(v\right).$$

The threshold and the limiting power function are obtained below with the help of numerical simulations.

Bayesian tests Suppose now that the parameter ϑ is a random variable with the *a priori* density $p(\theta)$, $\vartheta_1 \leq \theta < \beta$. This density is supposed to be continuous and positive. We consider two tests.

The first one (BT1) is based on the Bayesian estimator:

$$\widetilde{\phi}_n\big(X^{(n)}\big) = \mathbb{1}_{\left\{(\varphi_n^*)^{-1}\left(\widetilde{\vartheta}_n - \vartheta_1\right) > k_\varepsilon\right\}}.$$

As above, we have the convergence in distribution:

$$\left(\varphi_n^*\right)^{-1}\left(\widetilde{\vartheta}_n - \vartheta_1\right) \Rightarrow \zeta_+^* \triangleq \frac{\int_0^{+\infty} v Z^*\left(v\right) \mathrm{d}v}{\int_0^{+\infty} Z^*\left(v\right) \mathrm{d}v},$$

which allows us to chose the threshold such that $\widetilde{\phi}_n \in \mathscr{K}_\varepsilon$ as the solution of the equation

$$\mathbf{P}\left\{\zeta_+^* > k_\varepsilon\right\} = \varepsilon. \tag{10}$$

The power function of the BT1 has the following limit:

$$\beta\left(\widetilde{\phi}_n, u\right) = \mathbf{P}_{\vartheta_u}^{(n)}\left\{\left(\varphi_n^*\right)^{-1}\left(\widetilde{\vartheta}_n - \vartheta_u\right) + u > k_\varepsilon\right\} \to \mathbf{P}\left\{\zeta_u^* > k_\varepsilon - u\right\},$$

where the random variable ζ_u^* is given by

$$\zeta_u^* = \frac{\int_{-u}^{+\infty} v Z^*\left(v\right) \mathrm{d}v}{\int_{-u}^{+\infty} Z^*\left(v\right) \mathrm{d}v}.$$

Note that we can also derive another expression of the limiting power function of the BT1 as follows:

$$\beta\left(\widetilde{\phi}_n, u\right) = \mathbf{P}_{\vartheta_u}^{(n)}\left\{\left(\varphi_n^*\right)^{-1}\left(\widehat{\vartheta}_n - \vartheta_1\right) > k_\varepsilon\right\} \to \mathbf{P}\left\{\zeta_{u,+}^* > k_\varepsilon\right\},$$

where the random variable $\zeta_{u,+}^*$ is given by

$$\zeta_{u,+}^* = \frac{\int_0^{+\infty} v Z_u^*\left(v\right) \mathrm{d}v}{\int_0^{+\infty} Z_u^*\left(v\right) \mathrm{d}v}$$

The threshold and the limiting power function are obtained below with the help of numerical simulations.

The second test (BT2) minimizes the mean error. The likelihood ratio is

$$\widetilde{L}\big(X^{(n)}\big) = \int_{\vartheta_1}^{\beta} \frac{L_n\big(\theta, X^{(n)}\big)}{L_n\big(\vartheta_1, X^{(n)}\big)}\, p(\theta)\,\mathrm{d}\theta = \varphi_n^* \int_0^{(\varphi_n^*)^{-1}(\beta-\vartheta_1)} Z_{n,\vartheta_1}^*(v)\, p(\vartheta_1 + v\varphi_n^*)\,\mathrm{d}v.$$

Hence, we have the following limit:

$$\left(\varphi_n^*\right)^{-1}\widetilde{L}\big(X^{(n)}\big) \Rightarrow p\left(\vartheta_1\right)\int_0^{+\infty} \exp\left\{W\left(v\right) - \frac{v}{2}\right\} \mathrm{d}v.$$

Therefore, if we denote

$$R_n = \frac{(\varphi_n^*)^{-1}\,\widetilde{L}\big(X^{(n)}\big)}{p\,(\vartheta_1)}$$

and chose g_ε as solution of the equation

$$\mathbf{P}\left\{\int_0^{+\infty} \exp\left\{W\,(v) - \frac{v}{2}\right\}\mathrm{d}v > g_\varepsilon\right\} = \varepsilon,$$

the test $\mathbb{1}_{\{R_n > g_\varepsilon\}}$ belongs to the class $\mathscr{K}_\varepsilon$.

Numerical simulations Now, let us carry out some numerical simulations for the GLRT, the WT and the BT1. We take $r_n = n^{-0.25}$ and, in order to simplify the simulations, we take a function $\psi_n(t)$ depending neither on n nor on t. More precisely, we consider n independent trajectories $X_j^{(n)} = \big\{X_j^{(n)}(t),\ t \in [0,4]\big\}$, $j = 1,\ldots,n$, of an inhomogeneous Poisson process on the interval $[0,4]$ of intensity function

$$\lambda_\vartheta^{(n)}(t) = 1.5 + n^{-0.25}\,\mathbb{1}_{\{t>\vartheta\}}, \qquad 0 \le t \le 4,$$

with $\vartheta \in [2,4)$. So, denoting $\vartheta_1 = 2$ and

$$\varphi_n^* = \frac{\psi(\vartheta_1)}{n\,r_n^2} = \frac{1.5}{\sqrt{n}}\,,$$

we have (for $v \ge 0$)

$$\begin{aligned}\ln Z_{n,\vartheta_1}^*(v) &= \sum_{j=1}^n \int_{(\vartheta_1,\vartheta_1+v\varphi_n^*]} \ln\frac{1.5}{1.5+n^{-0.25}}\,X_j^{(n)}(\mathrm{d}t) + 1.5\,v\,n^{0.25}\\ &= \ln\frac{1.5}{1.5+n^{-0.25}}\sum_{j=1}^n \Big(X_j^{(n)}(\vartheta_1+v\varphi_n^*) - X_j^{(n)}(\vartheta_1)\Big) + 1.5\,v\,n^{0.25}.\end{aligned}$$

Some realizations of Z_{n,ϑ_1}^* can be found in Figure 1.

Recall that the threshold $h_\varepsilon = 1/\varepsilon$ of the GLRT is known explicitly. We obtain the threshold m_ε of the WT by numerically solving the equation (9), while the threshold k_ε of the BT1 is obtained from the equation (10) by means of numerical simulations of the random variable ζ_+^*. Some values of the thresholds m_ε and k_ε are given in Table 1.

Table 1 Thresholds of WT and BT1

ε	0.001	0.005	0.01	0.05	0.1	0.2
m_ε	30.336	20.686	14.886	7.282	4.531	2.236
k_ε	24.877	17.588	16.782	8.582	5.573	3.024

To illustrate the convergence of power functions of different tests to their limits, we present in Figure 2 the power functions for $n = 100$ ($r_n = 0.3162$) and $n = 300$ ($r_n = 0.2403$), as well as the limiting power functions. All these power functions are obtained by means of numerical simulations. Note that the values of u greater than $2(\varphi_n^*)^{-1}$ correspond to $\theta_u = \theta_1 + u\varphi_n^* > 4$, which means that there is no longer jump in intensity function on the interval $[0,4]$. This explains the fact that for $n = 100$, the power functions are constant for $u > 2(\varphi_{100}^*)^{-1} \approx 13.33$.

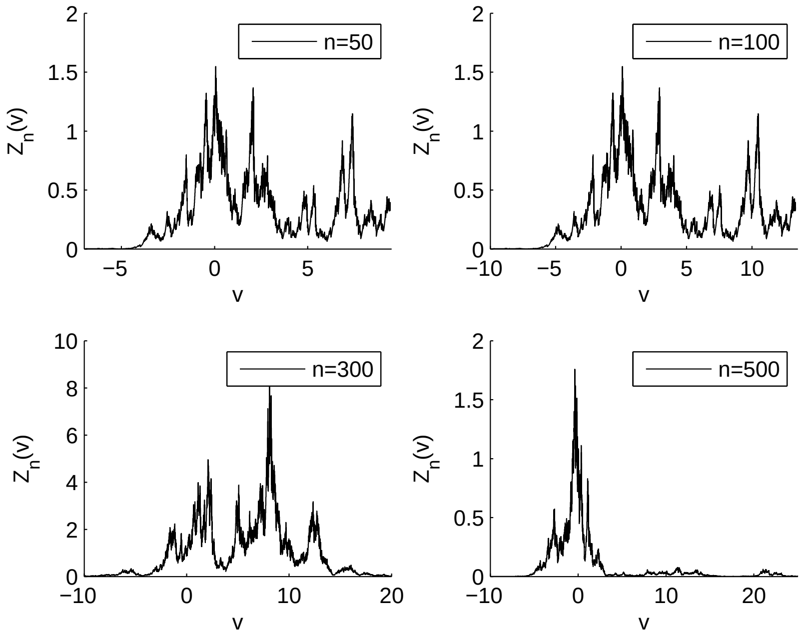

Fig. 1 Some realization of $Z^*_{n,\vartheta_1}(v)$

Comparison of the limiting power functions Let us fix an alternative $u_1 > 0$ and consider the testing problem with two simple hypotheses

$$\begin{aligned} \mathscr{H}_1 \quad &: \quad u = 0, \\ \mathscr{H}_2^{u_1} \quad &: \quad u = u_1. \end{aligned}$$

Remind that in this situation the most powerful test is the Neyman-Pearson test (N-PT). Of course, it is impossible to use the N-PT in our initial problem, because it depends on the value u_1 under alternative which is unknown. However, its power (considered as function of u_1) gives an upper bound (Neyman-Pearson envelope) for the power functions of all the tests. Therefore, it is interesting to compare the power functions of different tests not only one with another, but also with the power of the N-PT.

The N-PT is given by

$$\phi_n^*\left(X^{(n)}\right) = \mathbb{1}_{\left\{Z^*_{n,\vartheta_1}(u_1) > d_\varepsilon\right\}} + q_\varepsilon \mathbb{1}_{\left\{Z^*_{n,\vartheta_1}(u_1) = d_\varepsilon\right\}},$$

where d_ε and q_ε are solution of the equation

$$\mathbf{P}^{(n)}_{\vartheta_1}\left(Z^*_{n,\vartheta_1}(u_1) > d_\varepsilon\right) + q_\varepsilon \, \mathbf{P}^{(n)}_{\vartheta_1}\left(Z^*_{n,\vartheta_1}(u_1) = d_\varepsilon\right) = \varepsilon. \tag{11}$$

Recall that the likelihood ratio $Z^*_{n,\vartheta_1}(u_1)$ under hypothesis $\mathscr{H}_1$ converges to the following limit

$$Z^*_{n,\vartheta_1}(u_1) \Rightarrow Z^*(u_1) = \exp\left\{W(u_1) - \frac{u_1}{2}\right\}.$$

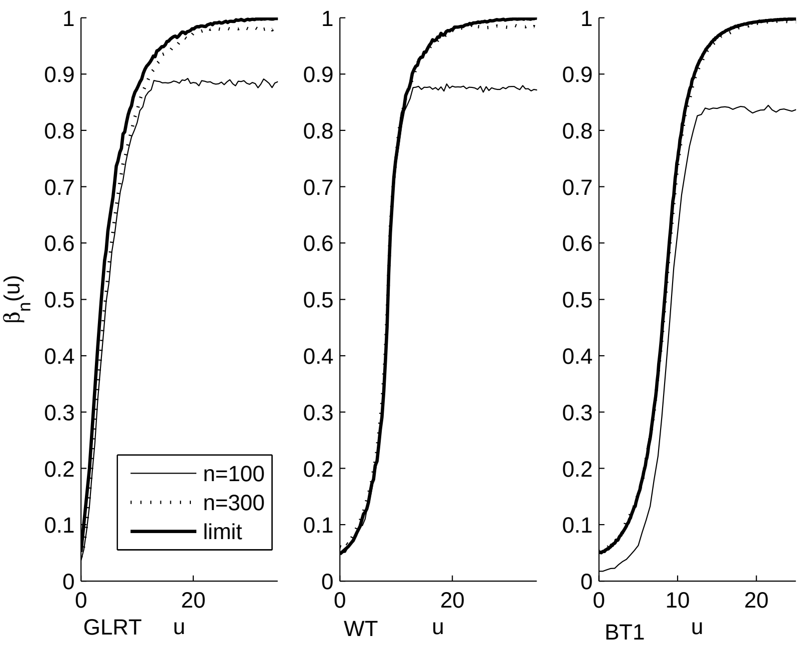

Fig. 2 Power functions of GLRT, WT and BT1

Hence, in the asymptotic setting, the equation (11) can be replaced by the equation

$$\mathbf{P}\big(Z^*(u_1) > d_\varepsilon\big) + q_\varepsilon\,\mathbf{P}\big(Z^*(u_1) = d_\varepsilon\big) = \varepsilon$$

and, since $Z^*(u_1)$ is a continuous random variable, we can put $q_\varepsilon = 0$ and find the threshold d_ε as the solution of the equation

$$\mathbf{P}\big(Z^*(u_1) > d_\varepsilon\big) = \varepsilon.$$

Note that

$$\mathbf{P}\big(Z^*(u_1) > d_\varepsilon\big) = \mathbf{P}\left(W(u_1) > \ln d_\varepsilon + \frac{u_1}{2}\right) = \mathbf{P}\left(\zeta > \frac{\ln d_\varepsilon + \frac{u_1}{2}}{\sqrt{u_1}}\right),$$

where $\zeta \sim \mathscr{N}(0,1)$. Therefore, denoting z_ε the quantile of order $1-\varepsilon$ of the standard Gaussian law $\big(\mathbf{P}(\zeta > z_\varepsilon) = \varepsilon\big)$, the threshold d_ε is given by

$$d_\varepsilon = e^{z_\varepsilon \sqrt{u_1} - \frac{u_1}{2}}.$$

Under alternative $\mathscr{H}_2^{u_1}$, we have

$$Z^*_{n,\vartheta_1}(u_1) = \frac{L_n\big(\vartheta_1 + u_1\varphi_n^*, X^{(n)}\big)}{L_n\big(\vartheta_1, X^{(n)}\big)} = \left(\frac{L_n\big(\vartheta_1 + u_1\varphi_n^* - u_1\varphi_n^*, X^{(n)}\big)}{L_n\big(\vartheta_1 + u_1\varphi_n^*, X^{(n)}\big)}\right)^{-1} \Rightarrow \big(Z^*(-u_1)\big)^{-1},$$

which allows us to obtain the limiting power of the N-PT as follows:

$$\begin{aligned}\beta\left(\phi_n^*\right) &= \mathbf{P}^{(n)}_{\vartheta_1+u_1\varphi_n^*}\left(Z^*_{n,\vartheta_1}(u_1) > d_\varepsilon\right)\\ &\to \mathbf{P}\left(\left(Z^*(-u_1)\right)^{-1} > d_\varepsilon\right) = \mathbf{P}\left(\exp\left\{-W(-u_1)+\frac{u_1}{2}\right\} > d_\varepsilon\right)\\ &= \mathbf{P}\left(W(u_1) > \ln d_\varepsilon - \frac{u_1}{2}\right) = \mathbf{P}\left(\zeta > \frac{\ln d_\varepsilon - \frac{u_1}{2}}{\sqrt{u_1}}\right) = \mathbf{P}\big(\zeta > z_\varepsilon - \sqrt{u_1}\big).\end{aligned}$$

So, the limiting Neyman-Pearson envelope is given by

$$\beta(u) = \mathbf{P}\big(\zeta > z_\varepsilon - \sqrt{u}\big) = 1 - \Phi\big(z_\varepsilon - \sqrt{u}\big),$$

where, as before, Φ is the distribution function of the standard Gaussian low.

The limiting power functions of the GLRT, of the WT and of the BT1 are obtained by means of numerical simulations and are presented in Figure 3 together with the limiting Neyman-Pearson envelope $\beta(u)$.

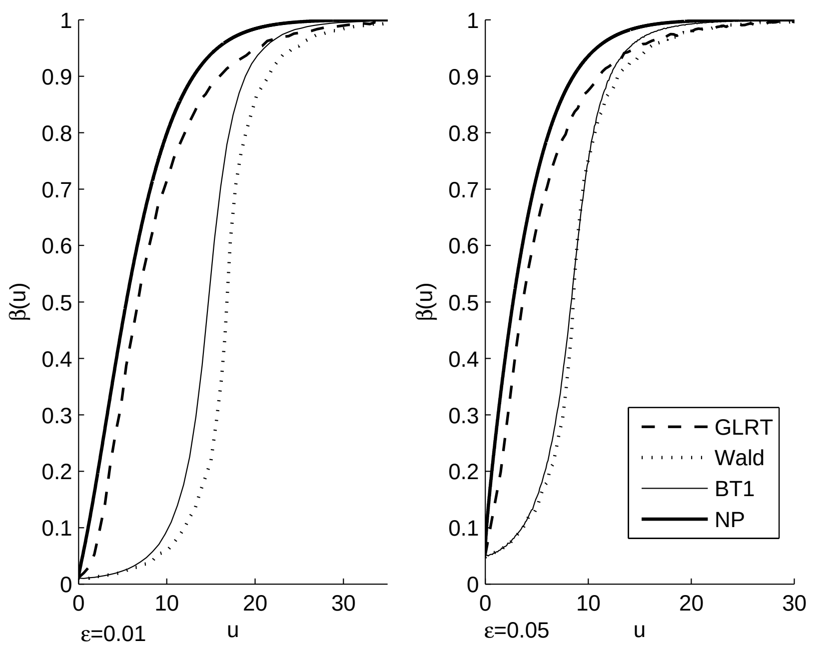

Fig. 3 Comparison of limiting power functions for $\varepsilon = 0.05$ and $\varepsilon = 0.4$

We can observe that the limiting power function of the GLRT is the closest to the limiting Neyman-Pearson envelope for small values of u, while the limiting power function of the BT1 is the one that tends to 1 (as u becomes large) the most quickly. We can also see that for $\varepsilon = 0.05$ the limiting power functions of the WT and of the BT1 are close (especially when u is small). Finally, we need to say that all these limiting power functions are perceptibly below the limiting Neyman-Pearson envelope, and that the choice of the asymptotically optimal test remains an open question.

6 Proofs of the lemmas

The proofs of Lemmas 2–4 in the case $r \neq 0$, as well as the proof of Lemma 5, are similar to the fixed jump size case and hence are omitted $\big($the interested reader can see, for example, [13, 14]$\big)$.

Proof of Lemma 2 in the case $r = 0$ First we study the convergence of 2-dimensional distributions. For this, consider the distribution of the vector $\big(Z_{n,\vartheta}(u_1), Z_{n,\vartheta}(u_2)\big)$ with some fixed $u_1, u_2 \in \mathbb{R}$. The characteristic function of the natural logarithm of this vector can be written as follows $\big($see, for example, [13]$\big)$:

$$
\begin{aligned}
\mathbf{E}_\vartheta^{(n)} & \exp\big(it_1 \ln Z_{n,\vartheta}(u_1) + it_2 \ln Z_{n,\vartheta}(u_2)\big) \\
&= \exp\bigg\{ n \int_0^\tau \bigg(\exp\Big\{ it_1 \ln \frac{\lambda_{\vartheta+u_1\varphi_n}^{(n)}(t)}{\lambda_\vartheta^{(n)}(t)} + it_2 \ln \frac{\lambda_{\vartheta+u_2\varphi_n}^{(n)}(t)}{\lambda_\vartheta^{(n)}(t)} \Big\} - 1 \\
&\qquad\qquad - it_1 \Big(\frac{\lambda_{\vartheta+u_1\varphi_n}^{(n)}(t)}{\lambda_\vartheta^{(n)}(t)} - 1 \Big) - it_2 \Big(\frac{\lambda_{\vartheta+u_2\varphi_n}^{(n)}(t)}{\lambda_\vartheta^{(n)}(t)} - 1 \Big) \bigg) \lambda_\vartheta^{(n)}(t)\, \mathrm{d}t \bigg\} \\
&= \exp\Big\{ A_{n,\vartheta}(u_1, u_2, t) \Big\}
\end{aligned}
$$

with an evident notation.

We will consider the case $u_2 > u_1 \geq 0$ only (the other cases can be treated in a similar way). In this case, we have

$$
\begin{aligned}
A_{n,\vartheta}(u_1, u_2, t) &= n \int_\vartheta^{\vartheta+u_1\varphi_n} \bigg(\exp\Big\{ (it_1 + it_2) \ln \frac{\psi_n(t)}{\psi_n(t) + r_n} \Big\} - 1 \\
&\qquad\qquad - (it_1 + it_2) \Big(\frac{\psi_n(t)}{\psi_n(t) + r_n} - 1 \Big) \bigg) \Big(\psi_n(t) + r_n \Big)\, \mathrm{d}t \\
&\quad + n \int_{\vartheta+u_1\varphi_n}^{\vartheta+u_2\varphi_n} \bigg(\exp\Big\{ it_2 \ln \frac{\psi_n(t)}{\psi_n(t) + r_n} \Big\} - 1 \\
&\qquad\qquad - it_2 \Big(\frac{\psi_n(t)}{\psi_n(t) + r_n} - 1 \Big) \bigg) \Big(\psi_n(t) + r_n \Big)\, \mathrm{d}t \\
&= nI_1 + nI_2
\end{aligned}
$$

with evident notations.

Using the mean value theorem for the integrals I_1 and I_2, it is possible to find some $s_n \in (\vartheta, \vartheta + u_1\varphi_n)$ and $v_n \in (\vartheta + u_1\varphi_n, \vartheta + u_2\varphi_n)$ such that

$$
nI_1 = \frac{u_1}{r_n^2} \Big(\exp\{ i(t_1 + t_2) \ln g_n(s_n) \} - 1 - i(t_1 + t_2)\big(g_n(s_n) - 1\big) \Big) \Big(\psi_n(s_n) + r_n \Big)
$$

and

$$
nI_2 = \frac{u_2 - u_1}{r_n^2} \Big(\exp\{ it_2 \ln g_n(v_n) \} - 1 - it_2\big(g_n(v_n) - 1\big) \Big) \Big(\psi_n(v_n) + r_n \Big),
$$

where we have denoted $g_n(t) = \frac{\psi_n(t)}{\psi_n(t)+r_n} = 1 - \frac{r_n}{\psi_n(t)+r_n}$.

As $s_n \to \vartheta$, using the condition **C3** we obtain $\lim_{n\to+\infty} \psi_n(s_n) = \psi(\vartheta)$. So,

$$nI_1 \sim \frac{u_1 \psi(\vartheta)}{r_n^2}\Big(\exp\{i(t_1+t_2)\ln g_n(s_n)\} - 1 - i(t_1+t_2)\big(g_n(s_n)-1\big)\Big).$$

As $r_n \to 0$ and $\ell \le \psi_n(t) + r_n \le L$, we have $g_n(s_n) - 1 = O(r_n) \to 0$. So, using Taylor expansion we get

$$\begin{aligned}\ln g_n(s_n) &= \ln\Big(1+\big(g_n(s_n)-1\big)\Big)\\ &= g_n(s_n) - 1 - \frac{1}{2}\big(g_n(s_n)-1\big)^2 + o\left(\frac{r_n^2}{(\psi_n(s_n)+r_n)^2}\right)\\ &= g_n(s_n) - 1 - \frac{1}{2}\big(g_n(s_n)-1\big)^2 + o(r_n^2).\end{aligned}$$

In particular, $\ln g_n(s_n) = O(r_n)$ and $\big(\ln g_n(s_n)\big)^2 = \big(g_n(s_n)-1\big)^2 + o(r_n^2)$.

Using Taylor expansion once more, we obtain

$$\exp\big(it\ln g_n(s_n)\big) = 1 + it\ln g_n(s_n) - \frac{t^2}{2}\big(\ln g_n(s_n)\big)^2 + o(r_n^2).$$

So,

$$\begin{aligned}nI_1 &\sim \frac{u_1\psi(\vartheta)}{r_n^2}\Big(-i(t_1+t_2)\frac{\big(g_n(s_n)-1\big)^2}{2} - \frac{(t_1+t_2)^2}{2}\big(g_n(s_n)-1\big)^2 + o(r_n^2)\Big)\\ &= \frac{u_1\psi(\vartheta)}{r_n^2}\Big(-\frac{i(t_1+t_2)r_n^2}{2(\psi(s_n)+r_n)^2} - \frac{(t_1+t_2)^2\, r_n^2}{2(\psi(s_n)+r_n)^2} + o(r_n^2)\Big)\\ &\to \frac{u_1}{\psi(\vartheta)}\Big(-\frac{i(t_1+t_2)}{2} - \frac{(t_1+t_2)^2}{2}\Big).\end{aligned}$$

Similarly, we can show that

$$nI_2 \to \frac{u_2-u_1}{\psi(\vartheta)}\Big(-\frac{it_2}{2} - \frac{t_2^2}{2}\Big),$$

and hence

$$\begin{aligned}&\mathbf{E}_\vartheta^{(n)} \exp\big(it_1\ln Z_{n,\vartheta}(u_1) + it_2\ln Z_{n,\vartheta}(u_2)\big)\\ &\qquad \to \exp\Big\{-\frac{u_2-u_1}{2\psi(\vartheta)}(it_2+t_2^2) - \frac{u_1}{2\psi(\vartheta)}\big(i(t_1+t_2)+(t_1+t_2)^2\big)\Big\}.\end{aligned} \tag{12}$$

For all $u > 0$, we introduce the σ-algebra $\mathscr{F}_u = \sigma\{W(v),\ 0 \le v \le u\}$ and write

$$\begin{aligned}&\mathbf{E}\exp\big(it_1\ln Z_\vartheta(u_1) + it_2\ln Z_\vartheta(u_2)\big)\\ &\quad = \mathbf{E}\bigg(\exp\big\{i(t_1+t_2)\ln Z_\vartheta(u_1)\big\}\mathbf{E}\Big(\exp\big\{it_2\big(\ln Z_\vartheta(u_2) - \ln Z_\vartheta(u_1)\big)\big\}\ \Big|\ \mathscr{F}_{u_1}\Big)\bigg)\\ &\quad = \exp\Big\{-\frac{(t_1+t_2)^2}{2\psi(\vartheta)}u_1 - \frac{i(t_1+t_2)}{2\psi(\vartheta)}u_1 - \frac{t_2^2}{2\psi(\vartheta)}(u_2-u_1) - \frac{it_2}{2\psi(\vartheta)}(u_2-u_1)\Big\}.\end{aligned}$$

Combining this with (12), we obtain the convergence of 2-dimensional distributions. The convergence of three and more dimensional distributions can be carried out in a similar way, and the uniformity with respect to ϑ is obvious. □

Proof of Lemma 3 in the case $r = 0$ We will consider the case $u_2 \geq u_1 \geq 0$ only (the other cases can be treated in a similar way). According to [14, Lemma 1.1.5], we have

$$\begin{aligned}
\mathbf{E}_\vartheta^{(n)} \left|Z_{n,\vartheta}^{1/2}(u_1) - Z_{n,\vartheta}^{1/2}(u_2)\right|^2 &\leq n \int_0^\tau \left(\sqrt{\lambda_{\vartheta+u_1\varphi_n}^{(n)}(t)} - \sqrt{\lambda_{\vartheta+u_2\varphi_n}^{(n)}(t)}\right)^2 \mathrm{d}t \\
&= n \int_{\vartheta+u_1\varphi_n}^{\vartheta+u_2\varphi_n} \left(\sqrt{\psi_n(t)+r_n} - \sqrt{\psi_n(t)}\right)^2 \mathrm{d}t \\
&= n \int_{\vartheta+u_1\varphi_n}^{\vartheta+u_2\varphi_n} \frac{r_n^2}{\left(\sqrt{\psi_n(t)+r_n} + \sqrt{\psi_n(t)}\right)^2} \mathrm{d}t.
\end{aligned}$$

As $\lambda_\vartheta^{(n)}$ is uniformly separated from zero, we have

$$\left(\sqrt{\psi_n(t)+r_n} + \sqrt{\psi_n(t)}\right)^2 \geq \left(\sqrt{\ell} + \sqrt{\ell}\right)^2 = 4\ell,$$

and hence

$$\mathbf{E}_\vartheta^{(n)} \left|Z_{n,\vartheta}^{1/2}(u_1) - Z_{n,\vartheta}^{1/2}(u_2)\right|^2 \leq n \int_{\vartheta+u_1\varphi_n}^{\vartheta+u_2\varphi_n} \frac{r_n^2}{4\ell} \mathrm{d}t = \frac{1}{4\ell} |u_1 - u_2|.$$

So, the required inequality holds with $C = \frac{1}{4\ell}$. □

Proof of Lemma 4 in the case $r = 0$ We will consider the case $u \geq 0$ only (the other case can be treated in a similar way). According to [14, Lemma 1.1.5], we have

$$\begin{aligned}
\mathbf{E}_\vartheta^{(n)} Z_{n,\vartheta}^{1/2}(u) &= \exp\left\{-\frac{n}{2}\int_0^\tau \left(\sqrt{\lambda_{\vartheta+u\varphi_n}^{(n)}(t)} - \sqrt{\lambda_\vartheta^{(n)}(t)}\right)^2 \mathrm{d}t\right\} \\
&= \exp\left\{-\frac{n}{2}\int_\vartheta^{\vartheta+u\varphi_n} \left(\sqrt{\psi_n(t)} - \sqrt{\psi_n(t)+r_n}\right)^2 \mathrm{d}t\right\} \\
&= \exp\left\{-\frac{n}{2}\int_\vartheta^{\vartheta+u\varphi_n} \frac{r_n^2}{\left(\sqrt{\psi_n(t)} + \sqrt{\psi_n(t)+r_n}\right)^2} \mathrm{d}t\right\}.
\end{aligned}$$

As $\lambda_\vartheta^{(n)}$ is uniformly bounded, we have

$$\left(\sqrt{\psi_n(t)+r_n} + \sqrt{\psi_n(t)}\right)^2 \leq \left(\sqrt{L} + \sqrt{L}\right)^2 = 4L,$$

and hence

$$\mathbf{E}_\vartheta^{(n)} Z_{n,\vartheta}^{1/2}(u) \leq \exp\left\{-\frac{n}{2}\int_\vartheta^{\vartheta+\varphi_n u} \frac{r_n^2}{4L} \mathrm{d}t\right\} = \exp\left\{-\frac{1}{8L}|u|\right\}.$$

So, the required inequality holds with $k_* = \frac{1}{8L}$. □

Proof of Lemma 6 Using Markov inequality, we get

$$\mathbf{P}_\vartheta^{(n)}\left(\left|\ln Z_{n,\vartheta}(u_1) - \ln Z_{n,\vartheta}(u_2)\right| > \varepsilon\right) \leq \frac{1}{\varepsilon^2}\mathbf{E}_\vartheta^{(n)}\left(\ln Z_{n,\vartheta}(u_1) - \ln Z_{n,\vartheta}(u_2)\right)^2.$$

First we consider the case $u_1, u_2 \geq 0$ (and say $u_2 \geq u_1$). In this case, we have

$$\begin{aligned}
\ln Z_{n,\vartheta}(u_2) - \ln Z_{n,\vartheta}(u_1) &= \sum_{j=1}^{n} \int_{\vartheta+u_1\varphi_n}^{\vartheta+u_2\varphi_n} \ln \frac{\psi_n(t)}{\psi_n(t)+r_n} \,\mathrm{d}X_j^{(n)}(t) + n\int_{\vartheta+u_1\varphi_n}^{\vartheta+u_2\varphi_n} r_n \,\mathrm{d}t \\
&= \sum_{j=1}^{n} \int_{\vartheta+u_1\varphi_n}^{\vartheta+u_2\varphi_n} \ln \frac{\psi_n(t)}{\psi_n(t)+r_n} \,\mathrm{d}Y_j^{(n)}(t) \\
&\quad + n\int_{\vartheta+u_1\varphi_n}^{\vartheta+u_2\varphi_n} \Big(\big(\psi_n(t)+r_n\big) \ln \frac{\psi_n(t)}{\psi_n(t)+r_n} + r_n \Big) \,\mathrm{d}t,
\end{aligned}$$

where $Y_j^{(n)}$ is the centered version of the process $X_j^{(n)}$.

Since the stochastic integrals with respect to $Y_j^{(n)}$, $j = 1, \dots, n$, are independent and has mean zero, we obtain

$$\begin{aligned}
\mathbf{E}_\vartheta^{(n)} \big(\ln Z_{n,\vartheta}(u_1) - \ln Z_{n,\vartheta}(u_2)\big)^2 &= n\,\mathbf{E}_\vartheta^{(n)} \left(\int_{\vartheta+u_1\varphi_n}^{\vartheta+u_2\varphi_n} \ln \frac{\psi_n(t)}{\psi_n(t)+r_n} \,\mathrm{d}Y_j^{(n)}(t) \right)^2 \\
&\quad + n^2 \left(\int_{\vartheta+u_1\varphi_n}^{\vartheta+u_2\varphi_n} \Big(\big(\psi_n(t)+r_n\big) \ln \frac{\psi_n(t)}{\psi_n(t)+r_n} + r_n \Big) \,\mathrm{d}t \right)^2 \\
&= E_1 + E_2
\end{aligned}$$

with obvious notations.

Using elementary inequalities $\ln(1+x) \leq x$ and $\ln(1+x) \geq x - x^2/2$ for $|x| < 1/2$, for sufficiently large values of n (such that $\frac{r_n}{\psi_n(t)+r_n} < \frac{r_n}{\ell} < 1/2$) we obtain

$$-\frac{r_n}{\psi_n(t)+r_n} - \frac{r_n^2}{2\big(\psi_n(t)+r_n\big)^2} \leq \ln \frac{\psi_n(t)}{\psi_n(t)+r_n} \leq -\frac{r_n}{\psi_n(t)+r_n}\,.$$

For E_1, if $r_n \leq 0$, we obtain

$$\begin{aligned}
E_1 &= n\int_{\vartheta+u_1\varphi_n}^{\vartheta+u_2\varphi_n} \Big(\ln \frac{\psi_n(t)}{\psi_n(t)+r_n} \Big)^2 \big(\psi_n(t)+r_n\big) \,\mathrm{d}t \\
&\leq n\int_{\vartheta+u_1\varphi_n}^{\vartheta+u_2\varphi_n} \frac{r_n^2}{\psi_n(t)+r_n} \,\mathrm{d}t \leq n\,\frac{(u_2-u_1)r_n^2\varphi_n}{\ell} = \frac{|u_1-u_2|}{\ell}\,.
\end{aligned}$$

As to the case $r_n \geq 0$, as $\frac{r_n}{\psi_n(t)+r_n} < 1/2$, we have

$$\begin{aligned}
E_1 &= n\int_{\vartheta+u_1\varphi_n}^{\vartheta+u_2\varphi_n} \Big(\ln \frac{\psi_n(t)}{\psi_n(t)+r_n} \Big)^2 \big(\psi_n(t)+r_n\big) \,\mathrm{d}t \\
&\leq n\int_{\vartheta+u_1\varphi_n}^{\vartheta+u_2\varphi_n} \left[\frac{r_n^2}{\psi_n(t)+r_n} + \frac{r_n^3}{(\psi_n(t)+r_n)^2} + \frac{r_n^4}{4(\psi_n(t)+r_n)^3} \right] \mathrm{d}t \\
&\leq n\int_{\vartheta+u_1\varphi_n}^{\vartheta+u_2\varphi_n} \frac{r_n^2}{\psi_n(t)+r_n} \left[1 + \frac{1}{2} + \frac{1}{16} \right] \mathrm{d}t \leq \frac{25\,|u_1-u_2|}{16\ell}\,.
\end{aligned}$$

For E_2, we have

$$-\frac{r_n^2}{2\ell} \leq -\frac{r_n^2}{2(\psi_n(t)+r_n)} \leq \big(\psi_n(t)+r_n\big) \ln \frac{\psi_n(t)}{\psi_n(t)+r_n} + r_n \leq 0,$$

and hence

$$\begin{aligned} E_2 &= n^2 \left(\int_{\vartheta+u_1\varphi_n}^{\vartheta+u_2\varphi_n} \left(\big(\psi_n(t)+r_n\big) \ln \frac{\psi_n(t)}{\psi_n(t)+r_n} + r_n \right) \mathrm{d}t \right)^2 \\ &\leq n^2 \left(\int_{\vartheta+u_1\varphi_n}^{\vartheta+u_2\varphi_n} \frac{r_n^2}{2\ell} \,\mathrm{d}t \right)^2 = \frac{(u_2-u_1)^2}{4\ell^2}. \end{aligned}$$

Thus, for sufficiently large values of n, we have

$$\mathbf{E}_\vartheta^{(n)} \big(\ln Z_{n,\vartheta}(u_1) - \ln Z_{n,\vartheta}(u_2)\big)^2 \leq \frac{25\,|u_1-u_2|}{16\ell} + \frac{(u_2-u_1)^2}{4\ell^2}.$$

In the case $u_1, u_2 \leq 0$, proceeding similarly, we obtain the same inequality.

Finally, in the case $u_1 u_2 < 0$ (say $u_1 < 0$ and $u_2 > 0$), we obtain

$$\begin{aligned} \mathbf{E}_\vartheta^{(n)} \big(\ln Z_{n,\vartheta}(u_1) - \ln Z_{n,\vartheta}(u_2)\big)^2 &\leq 2\,\mathbf{E}_\vartheta^{(n)} \big(\ln Z_{n,\vartheta}(u_1)\big)^2 + 2\,\mathbf{E}_\vartheta^{(n)} \big(\ln Z_{n,\vartheta}(u_2)\big)^2 \\ &\leq \frac{25\,|u_1|}{8\ell} + \frac{u_1^2}{2\ell^2} + \frac{25\,|u_2|}{8\ell} + \frac{u_2^2}{2\ell^2} \\ &= \frac{25}{8\ell}\big(|u_1|+|u_2|\big) + \frac{1}{2\ell^2}\big(u_1^2+u_2^2\big) \\ &\leq \frac{25\,|u_1-u_2|}{8\ell} + \frac{(u_2-u_1)^2}{\ell^2}. \end{aligned}$$

Note that this final inequality holds for all the three cases, and so

$$\mathbf{P}_\vartheta^{(n)} \big(\big|\ln Z_{n,\vartheta}(u_1) - \ln Z_{n,\vartheta}(u_2)\big| > \varepsilon\big) \leq \frac{25\,|u_1-u_2|}{8\varepsilon^2\ell} + \frac{(u_2-u_1)^2}{\varepsilon^2\ell^2}$$

for all $u_1, u_2 \in U_n$ and sufficiently large values of n. Hence,

$$\lim_{n\to+\infty} \sup_{|u_1-u_2|<h} \mathbf{P}_\vartheta^{(n)} \big(\big|\ln Z_{n,\vartheta}(u_1) - \ln Z_{n,\vartheta}(u_2)\big| > \varepsilon\big) \leq \frac{25h}{8\varepsilon^2\ell} + \frac{h^2}{\varepsilon^2\ell^2} \to 0$$

as $h \to 0$, and so, the lemma is proved. □

Proof of Lemma 7 We have

$$\mathbf{P}_\vartheta^{(n)} \left(\sup_{|u|>D} Z_{n,\vartheta}(u) > e^{-bD} \right) \leq \mathbf{P}_\vartheta^{(n)} \left(\sup_{u>D} Z_{n,\vartheta}(u) > e^{-bD} \right) + \mathbf{P}_\vartheta^{(n)} \left(\sup_{u<-D} Z_{n,\vartheta}(u) > e^{-bD} \right).$$

In order to estimate the first term, first let us note that the Markov process $Z_{n,\vartheta}(u)$, $u \geq 0$, is a martingale. Indeed, for any $v \geq u \geq 0$, using the representation (3) we can write

$$\begin{aligned} &\mathbf{E}\big(Z_{n,\vartheta}(v) \,\big|\, Z_{n,\vartheta}(u)\big) \\ &\quad= \mathbf{E}\left(\exp\left\{ \sum_{j=1}^n \int_{(\vartheta+u\varphi_n,\vartheta+v\varphi_n]} \ln \frac{\psi_n(t)}{\psi_n(t)+r_n}\, X_j^{(n)}(\mathrm{d}t) + \frac{v-u}{r_n} \right\} Z_{n,\vartheta}(u) \,\middle|\, Z_{n,\vartheta}(u) \right) \\ &\quad= Z_{n,\vartheta}(u) \exp\left\{ \frac{v-u}{r_n} \right\} \prod_{j=1}^n \mathbf{E} \exp\left\{ \int_{(\vartheta+u\varphi_n,\vartheta+v\varphi_n]} \ln \frac{\psi_n(t)}{\psi_n(t)+r_n}\, X_j^{(n)}(\mathrm{d}t) \right\} \\ &\quad= Z_{n,\vartheta}(u) \exp\left\{ \frac{v-u}{r_n} \right\} \exp\left\{ n \int_{\vartheta+u\varphi_n}^{\vartheta+v\varphi_n} \left(\frac{\psi_n(t)}{\psi_n(t)+r_n} - 1 \right) \big(\psi_n(t)+r_n\big)\,\mathrm{d}t \right\} \\ &\quad= Z_{n,\vartheta}(u) \exp\left\{ \frac{v-u}{r_n} \right\} \exp\left\{ -n \int_{\vartheta+u\varphi_n}^{\vartheta+v\varphi_n} r_n\,\mathrm{d}t \right\} \\ &\quad= Z_{n,\vartheta}(u) \exp\left\{ \frac{v-u}{r_n} \right\} \exp\big\{ -n(v-u)\varphi_n r_n \big\} = Z_{n,\vartheta}(u). \end{aligned}$$

Hence, the process $X(t) = Z_{n,\vartheta}^{1/2}(t+D)$, $t \geq 0$, is a supermartingale and, using the maximal inequality for positive supermartingales (see, for example, Revuz and Yor [16, Exercise 2.1.15]), we get

$$\mathbf{P}_{\vartheta}^{(n)}\left(\sup_{u>D} Z_{n,\vartheta}(u) > e^{-bD}\right) = \mathbf{P}_{\vartheta}^{(n)}\left(\sup_{t>0} X(t) > e^{-bD/2}\right) \leq e^{bD/2}\,\mathbf{E}X(0) = e^{bD/2}\,\mathbf{E}Z_{n,\vartheta}^{1/2}(D).$$

So, using Lemma 4, we can majorate the first term as

$$\mathbf{P}_{\vartheta}^{(n)}\left(\sup_{u>D} Z_{n,\vartheta}(u) > e^{-bD}\right) \leq e^{bD/2}\,e^{-k_* D} \leq e^{-bD},$$

where the last inequality is valid if $b \leq 2k_*/3$.

For the second term, in a similar manner (and under the same condition $b \leq 2k_*/3$) we obtain the bound

$$\mathbf{P}_{\vartheta}^{(n)}\left(\sup_{u<-D} Z_{n,\vartheta}(u) > e^{-bD}\right) \leq e^{-bD},$$

and so, the required inequality holds with $C = 2$ and any $b \in (0, 2k_*/3]$. □

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